PHYSICAL REVIEW E **72**, 046609 (2005)

Sound-field modeling in architectural acoustics by a transport theory: Application to street canyons

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(Received 1 March 2005; published 18 October 2005)

The transport theory of sound particles is applied to the sound field modeling in architectural acoustics. A theoretical description is proposed for empty enclosures with complex boundary conditions, including both specular and diffuse reflections. As an example, the model is applied to street canyons. Therefore, an asymptotic approach is proposed to reduce the transport equation to a diffusion equation defined by only one parameter, the diffusion coefficient. This coefficient is a function of the reflection law of the building façades, the ratio of specular and diffuse reflections, as well as the street width. The model is then compared to Monte Carlo simulations of the propagation of sound particles in complex enclosures. As expected by the asymptotic approach, the model is in agreement with numerical results, but mainly for small street width and very diffuse reflections. Finally, a discussion is proposed in the conclusion, on the model's capabilities.

DOI: 10.1103/PhysRevE.72.046609 PACS number(s): 43.55.+p, 43.20.+g, 05.60.Cd

I. INTRODUCTION

Sound field modeling in architectural acoustics has attracted considerable attention in the recent years. In both room acoustics and urban acoustics, several models have been derived to accurately predict the sound level and the sound decay, as well as many others acoustical parameters. The models take into account many effects, such as, specular reflection on boundaries, wall absorption, diffraction, diffusion and absorption by fitting objects, atmospheric attenuation, etc. More recently, research has been focussed on the effects of diffusely reflecting boundaries, showing that the scattering of sound by surface irregularities may have a significant influence on the sound field energy distribution and decay [1–3], as well as on the sound perception [4]. Consequently, it becomes difficult to develop a model that takes into account all the complex phenomena occurring in architectural acoustics.

Most of the time, analytical and numerical approaches based on the wave equation cannot be applied. Analytical solutions in terms of series expansions of normal modes [5–8], as well as numerical solutions based on finite and boundary elements methods [9] can only be achieved for simple enclosure shapes, uniform wall responses, and low frequencies (i.e., smaller than about one-third of the characteristic enclosure dimension) [10,11].

The classical theory of reverberation [12] widely applied in room acoustics is restricted to low absorption, quasicubic or ergodic enclosures without openings [13,14].

Standard image-source [1,2,15–17] and ray-tracing [18–22] methods are reduced to simple geometries, particularly ones that are polyhedric. By design, these methods are well adapted for specular reflections, but not for diffuse reflections. In addition, for complex shaped enclosures, the image-source and the ray-tracing methods imply extremely long calculation times, which limit the calculation of impulses responses to small order reflections. More recently, new techniques have been developed to introduce the effects of diffuse reflection in these standard methods, for instance, by combining a beam-tracing and a radiosity method [10,22–25] or by randomizing the direction of the reflected ray [26] in ray-tracing simulations. However, most of the time, the reflected energy distribution is assumed to follow Lambert's law (i.e., a cosinusoidal law), which has no physical reality [27]. Indeed, recent studies have shown that the boundary irregularities and protrusions may create complex reflection patterns, very different from Lambert's law [28].

Some statistical models have also been derived to predict the sound field in fitted rooms [29–33]. By using the concept of sound particles, and by considering a Markov or a diffusion process, authors have derived analytical expressions to predict the energy distribution in complex enclosures. However, diffuse reflections are taken into account by the same methods as discussed above.

Although several approaches may be considered to predict the sound field distribution in complex enclosures, only a few can take into account complex boundary conditions, including openings and diffuse reflections. Moreover, one can remark that each model can only be applied to one configuration. For example, investigations will be different in low halls [34], in corridors [35–37], in reverberation chambers [38,39], or in fitted rooms [40,41].

In this paper, a mathematical formulation is proposed to model the sound field encountered in architectural acoustics based on the concept of sound particles. Theoretically, all effects occurring during sound propagation (specular and diffuse reflections, absorption by walls and openings, diffusion

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by fitting objects, atmospheric attenuation, etc.) can be introduced analytically. For the moment, due to the mathematical complexity, this approach has been restricted to sound propagation in empty enclosures, with diffusely reflecting boundaries, and without atmospheric attenuation. The model is based on the concept of sound particles, particularly on the equivalence between the sound energy density and the single particle distribution function (SPDF) in a complex enclosure. Using this approach, the sound energy density in the enclosure is simply a solution of a transport equation with well-defined boundary conditions. As a first application, the model is applied to the sound propagation in a street canyon.

The concept of sound particles and the single particle distribution function are presented in Sec. II. The transport equation and the boundary equations are then derived in Sec. III, and applied to a street canyon in Sec. IV. Finally, the model is compared to Monte Carlo simulations in Sec. V.

II. SOUND PARTICLE FORMALISM

A. Energy representation of the reverberant field

The total acoustic field can be split in two parts, the direct sound field and the reverberant sound field. The direct sound field can be easily found. The reverberant sound field, which is the object of this paper, is the result of a complex energy mixing [1,42] due to multiple specular reflections, diffraction, and scattering occurring on the boundaries and in the enclosure. In the frequency range usually encounted in architectural acoustics (i.e., 100–5000 Hz), the effects of phase cancellation and addition that produce the classical interferences are averaged [43] and sound sources may be considered uncorrelated. Thus, the total energy at a receiver is equal to the sum of each energy contribution. Accordingly, a reverberant sound field can be modelled by using an energy approach. Despite neglecting the wave behavior of the sound field, energy models can provide satisfactory results such as reverberation times and sound attenuation [10].

Most of the energy models used in architectural acoustics are based on geometrical acoustic assumptions, which assume that the sound propagation may be represented by sound rays [12] propagating along straight lines between two collisions with the enclosure boundaries and the obstacles in the medium. Then, the total sound energy at a receiver located in an enclosure is calculated by summing all the sound rays crossing the receiver volume. In order to take the geometrical spreading due to the expansion of the wave fronts into account, the energy of each sound ray decreases inversely as the square of the distance from the source.

Another approach is to consider a sound ray as the path of an infinitesimal entity, named *sound particle* or *phonon* [19,44,45], with a constant energy. In this way, as discussed by Joyce [44,46], geometrical acoustics is a special case of the classical-particle dynamic. The main interest of this approach, is that the reverberant sound field may be seen as a gas of sound particles. Therefore, the classical formulation of gas theory can then be applied to acoustical problems.

B. Sound particle concept

A sound particle is defined as a classical point particle by its elementary energy e, its position $\mathbf{x}$, and its velocity $\mathbf{v}$,

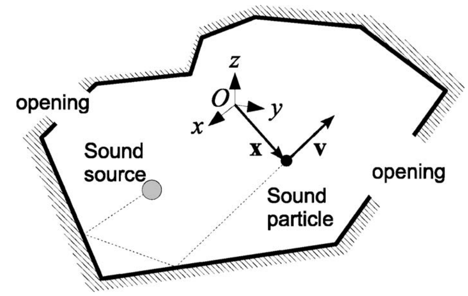

FIG. 1. Sound particle propagation in a semiopened and empty enclosure. $\mathbf{x}$ and $\mathbf{v}$ are the position and the velocity of the sound particle, respectively.

whose norm $\|\mathbf{v}\|$ is equal to the sound velocity c (Fig. 1). Space and velocity domains are defined, respectively, on the sets $X \in \mathbb{R}^3$ and $V \in [-c,c] \times [-c,c] \times [-c,c]$. Particle motion is characterized by its position vector $\mathbf{x}$ and its velocity vector $\mathbf{v}$,

$$\mathbf{x} = (x,y,z) \quad \text{with } (x,y,z) \in X, \tag{1}$$

$$\mathbf{v} = (u,v,w) \quad \text{with } (u,v,w) \in V.$$

The boundary of the subset X is noted ∂X. It corresponds to the geometrical boundary of the propagation medium.

Interactions and collisions between particles are neglected. A phonon obeys the laws of classical mechanics based on the Hamilton stationary action principle [47] and, in this case, undergoes a straight line until its impact with the obstacles or walls of the enclosure. During a collision with a scattering object or with a surface, the velocity direction is instantaneously deflected (Fig. 1).

C. The single particle distribution function

A particular state of a system with N particles is described by $3N$ coordinates of position $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\ldots,\mathbf{x}_N)$ and $3N$ coordinates of velocity $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\ldots,\mathbf{v}_N)$. This state is represented by a point in a $6N$ dimension phase space Γ [48]. If, at a given time t, it was possible to know exactly each particle position and velocity, it would be then possible to predict their position and velocity at a time $t+dt$. However, the number of sound particles being very large, the practical implementation is very difficult. As a consequence, the problem must be approached in a probabilistic way.

It is assumed that mutual particle interactions are not taken into account. Moreover, the large energy mixing due to scattering effects is sufficient to satisfy the ergodicity criterion. Hence, the description of the N particles system can be reduced to the knowledge of a fictive single particle system [49], in a six dimensional phase space μ, defined by the single particle distribution function $f(\mathbf{x},\mathbf{v},t)$ [49,50]. This function characterizes the statistical behavior of a sound particle, and

$$f(\mathbf{x},\mathbf{v},t)d\mathbf{x}\, d\mathbf{v}, \quad \mathbf{x} \in X, \quad \mathbf{v} \in V, \tag{2}$$

represents the amount of particles, at time t, with velocity $\mathbf{v}$ to within about $d\mathbf{v}$, in an elementary volume $d\mathbf{x}$ located at $\mathbf{x}$ (Fig. 2).

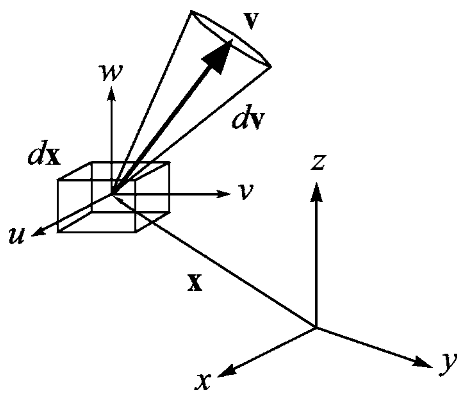

FIG. 2. Representation of $d\mathbf{x}$ and $d\mathbf{v}$.

The local density of sound particles is defined by the integration of the SPDF over the velocity space,

$$n(\mathbf{x},t) = \int_{\mathbf{v}} f(\mathbf{x},\mathbf{v},t)d\mathbf{v}, \quad \mathbf{x} \in X, \quad \mathbf{v} \in V. \tag{3}$$

The sound energy density $w(\mathbf{x},t)$ is obtained by multiplying the local density by the elementary energy of the sound particles

$$w(\mathbf{x},t) = en(\mathbf{x},t), \quad \mathbf{x} \in X, \quad \mathbf{v} \in V. \tag{4}$$

The local particle flow, which represents the number of particles crossing a unit area per unit of time at $\mathbf{x}$ and time t, is the integration over the velocity space of the product of the velocity and the SPDF, defined by

$$\mathbf{J}(\mathbf{x},t) = \int_{\mathbf{v}} \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v}, \quad \mathbf{x} \in X, \quad \mathbf{v} \in V. \tag{5}$$

In the same way, the local sound energy flow at $\mathbf{x}$ and time t is given by

$$\mathbf{E}(\mathbf{x},t) = e\int_{\mathbf{v}} \mathbf{v}f(\mathbf{x},\mathbf{v},t)d\mathbf{v}, \quad \mathbf{x} \in X, \quad \mathbf{v} \in V. \tag{6}$$

III. SOUND PARTICLE TRANSPORT THEORY

A. Transport equation

Since collisions of phonons take place on the boundaries only, the spatial and the temporal evolution of the sound particle density in the enclosure is similar to the evolution of the molecular density in a rarefied gas or Knudsen gas [51]. The main equation of the model, can be derived from transport theory [49] to give

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = 0, \quad \mathbf{x} \in X, \quad \mathbf{v} \in V, \tag{7}$$

where $\nabla_{\mathbf{x}}$ represents the spatial derivative. This is the transport equation of free molecular flow [52], also called Liouville equation, expressing the variation of the particle density during dt according to a transport phenomenon [operator $\mathbf{v}\cdot\nabla_{\mathbf{x}}$ acting on $f(\mathbf{x},\mathbf{v},t)$].

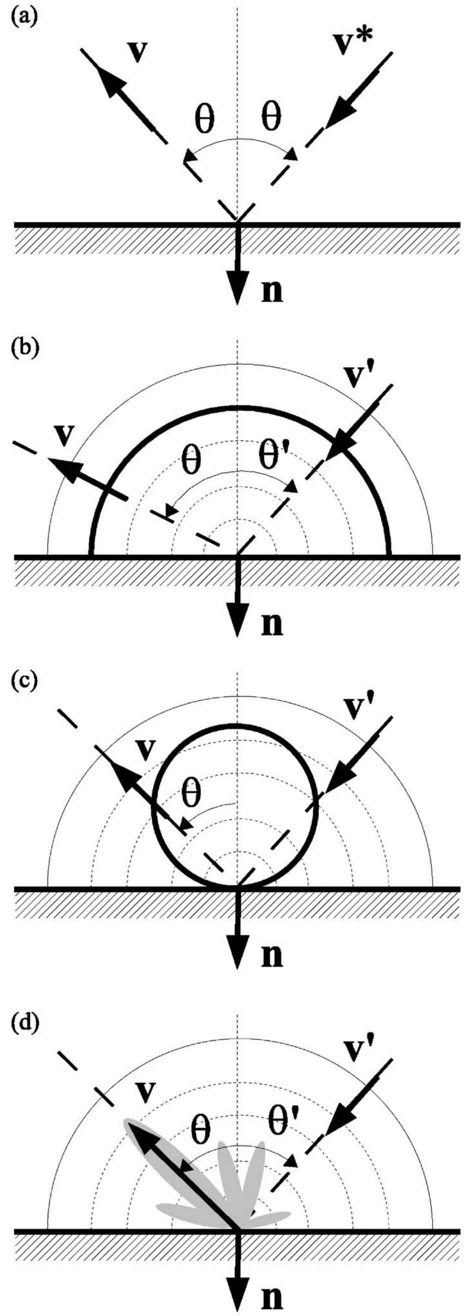

FIG. 3. Schematic representation of two-dimensional reflection laws $\mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}') \equiv \mathcal{R}(\mathbf{x},\theta,\theta')$. θ and θ' are the reflected and incident angles, respectively. $\mathbf{v}$ and $\mathbf{v}'$ are the reflected and incident velocity, respectively. $\mathbf{v}^*$ is the particular incident velocity combined with a specular reflection. (a) Specular reflection ($\theta = \theta'$), (b) uniform reflection law [i.e., $\mathcal{R}(\mathbf{x},\theta,\theta')$ is constant], (c) Lambert's law [i.e., $\mathcal{R}(\mathbf{x},\theta,\theta')$ is proportional to cos θ], (d) complex reflection law. $\mathbf{n}$ is the outgoing normal to the façades.

B. Boundary conditions

Typically, surfaces have considerable irregularities due to window recesses, decorative structures, doorways, ledges, seats, columns, etc. According to the frequency and the size of wall protrusions, reflections may or may not be specular. Whenever wavelengths are much smaller than the size of irregularities or than the dimensions of the enclosure, reflections are specular. Nevertheless, when the wavelength is of the order of wall protrusion size, scattering effects are numerous, producing a diffuse reflection. In architectural acoustics, the wide frequency range encountered implies most of the time a mix of both reflections. Moreover, wall absorption must be also considered.

First of all, it is convenient to introduce the outward unit normal $\mathbf{n}$ to the boundary in $\mathbf{x} \in \partial X$ (Fig. 3). Let us define by $\mu^{\pm}$ the points $(\mathbf{x},\mathbf{v})$ of the phase space μ that satisfy the relation

$$\mu^{\pm}:\{(\mathbf{x},\mathbf{v})/\mathbf{x} \in \partial X, 0 < \pm\, \mathbf{n}\cdot\mathbf{v}\}. \quad (8)$$

The restriction of f to the set μ^{+} (μ^{-}, respectively) is noted below f^{+} (f^{-}, respectively).

1. Wall absorption

The absorption coefficient $\alpha(\mathbf{x})$ is defined for $\mathbf{x} \in \partial X$, as the probability that a sound particle is absorbed by the boundary. It varies from 0, for total reflection, to 1 for total absorption.

2. The accommodation coefficient

In order to take both reflections into account, the accommodation coefficient $d(\mathbf{x})$ (for $\mathbf{x} \in \partial X$) can be introduced. This coefficient is representative of the boundary morphology, and expresses the part of nonspecular (i.e., diffuse) and specular reflection. By definition, it varies from 0 for a diffuse reflection, to 1 for a full specular reflection. This accommodation coefficient can be in keeping with the usual diffusion coefficient $\delta = 1 - d$, defined in room acoustics [2,9,53,54].

3. Specular reflection

The specular reflection can be considered in a deterministic way. Indeed, the knowledge of the incidence angle θ' gives the angle of reflection according to the Snell-Descartes laws. The relationship between the reflected velocity $\mathbf{v}$ and the incident velocity $\mathbf{v}^*$ of a sound particle at the boundary with normal $\mathbf{n}$ is then given by [Fig. 3(a)]

$$\mathbf{v}^* = \mathbf{v} - 2(\mathbf{n}\cdot\mathbf{v})\mathbf{n}. \quad (9)$$

4. Diffuse reflection

As introduced above, diffuse reflections may have significant effects on the sound field decay and distribution in an enclosure. In order to take these effects into account, some authors have proposed to use the uniform reflection law [Fig. 3(b)] which distributes the sound energy in all directions with the same probability, or Lambert's law [Fig. 3(c)] favoring the normal direction. However, although both laws are used in computer models, there is no real evidence of the physical reality of these laws in architectural acoustics.

In the present approach, the scattering effects on the boundaries are modelled by considering an arbitrary reflection law $\mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}')$ for $\mathbf{x} \in \partial X$ [Fig. 3(d)]. This reflection law represents the probability that a sound particle with an incident velocity $\mathbf{v}'$ leaves the boundary at position $\mathbf{x} \in \partial X$ with a velocity $\mathbf{v}$ after reflection. This function is normalized to unity,

$$\int_{\Gamma^+} \mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}')d\mathbf{v}' = 1, \quad \mathbf{x} \in \partial X, \quad \mathbf{v} \in V, \quad (10)$$

and must satisfy the reciprocity relation,

$$\int_{\Gamma^-} \mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}')d\mathbf{v} = 1, \quad \mathbf{x} \in \partial X, \quad \mathbf{v} \in V, \quad (11)$$

which expresses the flow conservation on the boundaries. This reflection law is equivalent to the scattering kernel defined in gas theory [51,55].

5. Flow conservation at boundaries

The boundary conditions express the reflected particle flow as a function of the incident particle flow. By considering the part of specularly and diffusely reflected sound particles, the flow conservation is written

$$\underbrace{|\mathbf{n}\cdot\mathbf{v}|f^-(\mathbf{x},\mathbf{v},t)}_{\text{reflected flow}} = [1-\alpha(\mathbf{x})]\Big(\underbrace{d(\mathbf{x})|\mathbf{n}\cdot\mathbf{v}^*|f^+(\mathbf{x},\mathbf{v}^*,t)}_{\text{incident flow: specular reflection}} + \underbrace{[1-d(\mathbf{x})]\int_{\Gamma^+}\mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}')|\mathbf{n}\cdot\mathbf{v}'|f^+(\mathbf{x},\mathbf{v}',t)d\mathbf{v}'}_{\text{incident flow: diffuse reflection}}, \quad (12)$$

for $\mathbf{x} \in \partial X$ and $\mathbf{v} \in \Gamma^-$. The left-hand side of Eq. (12) represents the reflected flow. The right-hand side, weighted by the reflection coefficient $[1-\alpha(\mathbf{x})]$, represents the specular flow (first term) and the diffuse flow (second term).

C. Application

The application of the transport model to a specific problem, like sound propagation in rooms or streets, requires solving the transport equation (7) with the boundary conditions defined by Eq. (12). However, presently there is still no exact analytical solution for such a system of equations, even for simple geometries. Conversely, asymptotic solutions may be found in some cases, for example, when a specific dimension is larger than the others, like in corridors, street canyons, and low halls.

Thus, the main difficulty lies first, in the choice of the assumptions that are needed to simplify the geometry of the problem, and second, in the choice of the asymptotic approach that must be used in order to give simple and adapted solutions. In the related literature, some asymptotic approaches are proposed. In particular, one can mention the recent studies by Babovsky *et al.* [56] and Börgers *et al.* [57], applied to the diffusion approximation for a Knudsen gas in a thin domain with accommodation on the boundary. Such approaches seem well adapted to sound propagation modeling in corridors and in streets, for which the length is larger than the width or the height. As an example, an application to sound propagation in a street canyon is detailed in the next section.

IV. APPLICATION TO SOUND PROPAGATION IN A STREET CANYON

A. Introduction

Noise is a major problem for people living in urban areas. Consequently, many mathematical and numerical models have been developed to predict sound propagation in streets, using, for example, the image-source methods, the modal approaches, or the classical theory of reverberation. However, because of the complexity of the façade effects, these models are still not satisfactory. Conversely, the use of the transport model seems to be a solution for sound propagation modeling in urban areas. The study of street canyons, instead of "regular" streets, is not really a restriction. Noise occurs in strongly built-up urban areas, where the sound activity is significant, like in the city center. In such places, streets and boulevards are very numerous and buildings are very high. As a first approximation, the propagation medium can then be compared to a network of street canyons.

B. The street canyon

Let us now consider a street canyon, of width $2L$, height l_y and length l_x [Fig. 4(a)]. It is assumed that the width is smaller than the size of the building façades (i.e., $2L \ll l_y$ and $2L \ll l_x$) and, thus, the street may be characterized by an infinite length and height. In addition, recent studies have shown that typical urban vehicles, like cars and motorcycles, may be modelled as point sound sources, located a few centimeters above the ground [58]. For this reason, in the present approach, the sound source S is assumed to be on the pavement. To simplify the following developments, the sound source (defined by a sound power P) is located in the middle of the street section. The pavement is assumed to be a perfectly reflecting plane (i.e., a mirror plane). Conversely, it is assumed that the building façades produce diffuse and specular reflections. In both cases, sound absorption is assumed very low [59,60] and can be neglected.

According to the image-source theory, the pavement is a mirror plane for the sound source and the building façades. It is assumed that the sound source S and its image are merged into one virtual source S' with a sound power $2P$. Then, the problem of sound propagation in the street canyon may be compared to the propagation of sound between two infinite and parallel diffusely reflecting surfaces [Fig. 4(b)]. The transport problem is then reduced to the propagation of sound particles in a *narrow* propagation medium, delimited by two diffusely parallel planes.

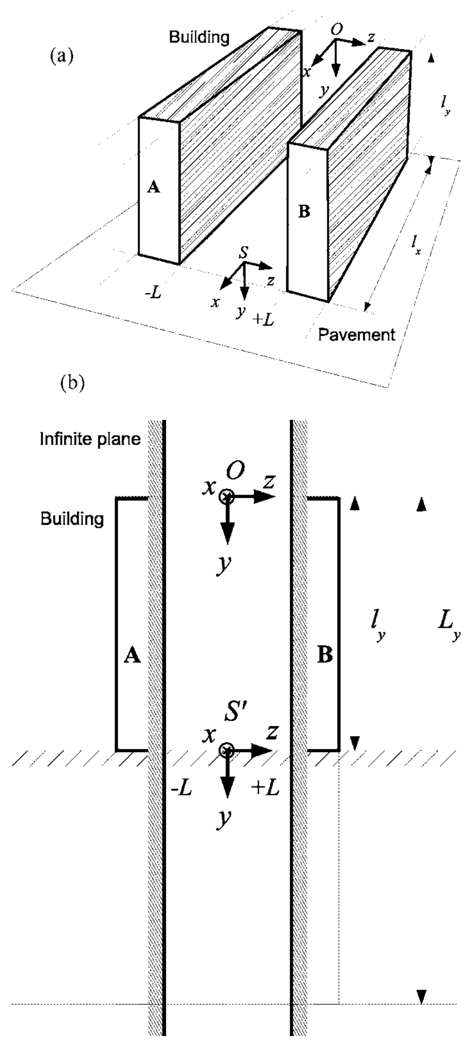

FIG. 4. Propagation medium modeling, (a) street with a sound source S standing on the ground; (b) street modeling as two infinite and parallel planes with a virtual source S'. The street width is $2L$.

Following Sec. III B, boundary reflections on the parallel planes (i.e., the building façades) are expressed by the accommodation coefficient d and the diffuse reflection law $\mathcal{R}(\mathbf{x},\mathbf{v},\mathbf{v}')$. Although, recent work has been carried out to characterize diffusely reflecting surfaces [61], there is still no satisfactory and accurate results. At the present time, analytical expressions of reflection laws are not available, apart from the uniform reflection law and Lambert's law. In this paper, the reflection law is considered a function of the normal component of the velocity $|w|$ (Fig. 5), and is independent of the space variable $\mathbf{x}$,

$$\mathcal{R}(\mathbf{v},\mathbf{v}') \propto |w|^k. \tag{13}$$

It may be noted that for $k=0$ and $k=1$, this expression gives the uniform reflection law and Lambert's law, respectively. The variation of the parameter k allows to change the behavior of the sound reflection from a uniform diffusion to extremely concentrated diffusion around the normal to the

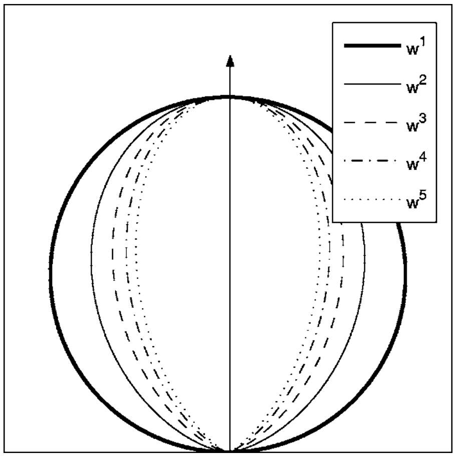

FIG. 5. Two-dimensional polar diagrams for $|w|^k$ reflection laws with $k=1, 2,\ldots,5$.

façade. Last, in this paper, the diffuse reflection law and the accommodation coefficient are assumed to be uniform and equal on both building façades.

C. Asymptotic development

The asymptotic approach detailed in this section is taken from the recent study by Börgers *et al.* [57] and is based on a diffusion approximation resulting from simple geometrical considerations. Figure 6 shows the propagation of a sound particle between two planes. At each collision with one of the planes, the direction of the velocity vector is changed. If the mean free path of the sound particle is large, this change of direction appears after a long propagation time. This is in agreement with a transport process. However, the smaller the distance between the planes, the smaller is the mean free path and faster is the change of direction. Thus, in a very narrow medium, the particle will experience more and more collisions leading to a rapid variation of the velocity vector direction (Fig. 7). This last behavior is characteristic of a diffusion process. Hence, the diffusion process may be considered as the limit of the transport process in a narrow medium. The aim of the asymptotic development is then to develop the transport equation in order to reach a diffusion equation.

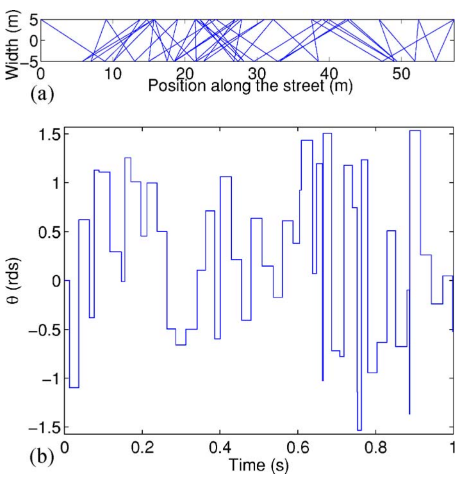

FIG. 6. (Color online) (a) 2D propagation of a single particle between two parallel planes (i.e., $z=-5$ and $z=+5$) with $2L=10$ m; (b) temporal variation of the angle θ between the plane $z=-5$ and the velocity vector.

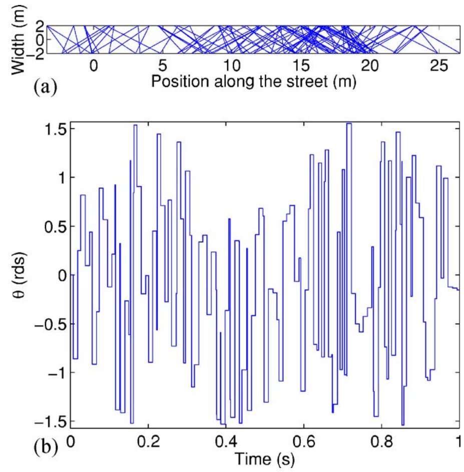

FIG. 7. (Color online) (a) 2D propagation of a single particle between two parallel planes (i.e., $z=-2$ and $z=+2$) with $2L=4$ m; (b) temporal variation of the angle θ between the plane $z=-2$ and the velocity vector.

1. Geometry

In this section, the asymptotical approach is developed in two dimensions in order to simplify the mathematical expressions. However, it can be easily extended to three dimensions. Finally results will be given for both cases.

In two dimensions, the problem is reduced to the propagation of sound particles between two parallel lines (Fig. 8). Mathematically, the space domain is defined on the set $Y \in \mathbb{R}^2$ subdivided in two sets X and Z such that $Y=X\times Z$, with $X\in\mathbb{R}$ and $Z\in\,]-L,L[$. The propagation of a sound particle is entirely defined by the position vector $\mathbf{x}$ and the velocity vector $\mathbf{v}$, with

$$\mathbf{x}=(x,z) \quad \text{with } x\in X, z\in Z,$$

$$\mathbf{v}=(v,w) \quad \text{with } (v,w)\in[-c,c]\times[-c,c]=V. \tag{14}$$

The boundary of the subset Z is noted ∂Z. Therefore, the boundary of the set Y noted $\partial Y=\partial Z\times X$, represents the geometrical boundaries of the propagation medium, that is the two lines at $z=+L$ and $z=-L$.

2. Change of variables

In order to reach a diffusion process, it is necessary to bring the building façades closer, or equivalently, to rescale the time variable to increase the number of collisions by unit time. This can be done by introducing a small parameter ε in the transport equation and afterwards, to study its behavior as ε tends to zero. The time variable t is then rescaled as t/ε, while the space coordinate z is expanded as εz. Letting ε

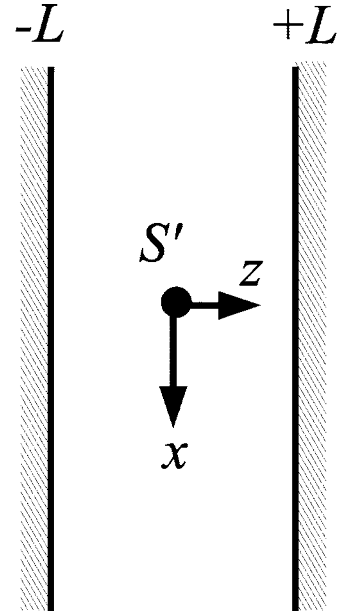

FIG. 8. Two-dimensional representation of the street, with a sound source S' located in the middle of the street.

tend to zero allows to simultaneously obtain a time expansion and a spatial reduction, leading, respectively, to a longer observation of the particle transport and to an increase in the number of collisions. The modified transport equation is now written

$$\frac{\partial f_\varepsilon}{\partial t} + \frac{1}{\varepsilon} v \frac{\partial f_\varepsilon}{\partial x} + \frac{w}{\varepsilon^2}\frac{\partial f_\varepsilon}{\partial z} = 0, \tag{15}$$

for $(x,z) \in Y$, $(v,w) \in V$ and where $f_\varepsilon(x,z,v,w,t) = f(x,\varepsilon z,v,w,t/\varepsilon)$.

The boundary conditions are unchanged, since they act on the velocity variables only, and not on z or t,

$$|w| f_\varepsilon^-(x,z,v,w,t) = d|w| f_\varepsilon^+(x,z,v^*,w^*,t) + (1-d) \times \int_{\Gamma^+} \mathcal{R}(z,\mathbf{v},\mathbf{v}') \times |w'| f_\varepsilon^+(x,z,v',w',t) dv' dw' \tag{16}$$

for $(x,z) \in \partial Y$, $\mathbf{v} \in \Gamma^-$, and where the absorption by the pavement and the building façades has been neglected. The initial condition ensures the conservation of the number of sound particles in the medium,

$$\iint f_\varepsilon(\mathbf{x},\mathbf{v},t) d\mathbf{x}\, d\mathbf{v} = \iint F(\mathbf{x},\mathbf{v},0) d\mathbf{x}\, d\mathbf{v} = N. \tag{17}$$

3. Variable separation

Considering the last change of variables, the transport equation is now written as

$$-\varepsilon\left(\varepsilon \frac{\partial f_\varepsilon}{\partial t} + v \frac{\partial f_\varepsilon}{\partial x}\right) = w \frac{\partial f_\varepsilon}{\partial z}. \tag{18}$$

The next step of the mathematical development is to express the distribution function as the product of two functions. We consider that the function $f_\varepsilon(x,z,v,w,t)$ converges to a new function $\mathcal{F}(x,z,v,w,t)$ when ε tends to zero, such as

$$\lim_{\varepsilon\to 0} -\varepsilon\left(\varepsilon \frac{\partial f_\varepsilon}{\partial t} + v \frac{\partial f_\varepsilon}{\partial x}\right) = \lim_{\varepsilon\to 0} w \frac{\partial f_\varepsilon}{\partial z} \tag{19a}$$

$$= w \frac{\partial}{\partial z} \mathcal{F}(x,z,v,w,t) \tag{19b}$$

$$= 0. \tag{19c}$$

In Eq. (19a), variables x and t are only parameters. Thus, it is possible to realize a separation of variables and, afterwards, to express the function $\mathcal{F}(x,z,v,w,t)$ as the product of two functions,

$$\mathcal{F}(x,z,v,w,t) = q(x,t) \times \phi(z,v,w), \tag{20}$$

where $\phi(z,v,w)$ is positive, normalized to unity,

$$\int\int\int \phi(z,v,w) dz\, dv\, dw = 1, \tag{21}$$

and from (19a), verifies

$$w \frac{\partial}{\partial z} \phi(z,v,w) = 0. \tag{22}$$

This last result means that the particle density is constant between the planes (i.e., along the z direction). The sound particle density varies along the x direction only, according to the function $q(x,t)$. Following this approach, the sound energy in a street canyon is constant along a transversal line of the street. This is in agreement with recent experimental results [67] and justifies a common hypothesis made in most cases of sound propagation in long spaces, like corridors and streets [59].

4. The diffusion equation

As suggested before, the diffusion process is a "macroscopic" view of the particle transport problem when the number of collisions per unit of time increases. In order to reach a diffusion process, it is necessary to consider "macroscopic" variables. As an example, the average local density of sound particles may be considered. It corresponds to the integration of the distribution function over the velocity space (v,w) and the width z, and is simply equal to $q(x,t)$ [from Eqs. (20) and (21)]:

$$\int\int\int f_\varepsilon(x,z,v,w,t) dz\, dv\, dw = q(x,t). \tag{23}$$

In order to express the transport problem in a macroscopic view, the transport equation can be integrated in the same way, to give

$$\iiint \frac{\partial f_\varepsilon}{\partial t} dz\, dv\, dw + \frac{1}{\varepsilon} \iiint v \frac{\partial f_\varepsilon}{\partial x} dz\, dv\, dw$$
$$= -\frac{1}{\varepsilon^2} \iiint w \frac{\partial f_\varepsilon}{\partial z} dz\, dv\, dw. \tag{24}$$

Taking the limit of Eq. (24), as ε tends to zero, yields

$$\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iiint v f_\varepsilon dz\, dv\, dw = 0, \tag{25}$$

since the right-hand side is equal to zero due to the boundary conditions. To determine the second term on the left-hand side, a new function $D^*(z,v,w)$ is introduced,

$$-w \frac{\partial}{\partial z} D^*(z,v,w) = v, \tag{26}$$

where the symbol $*$ represents the conjugate of the function $D(z,v,w)$, defined by

$$w \frac{\partial}{\partial z} D(z,v,w) = v \phi(z,v,w). \tag{27}$$

This function $D(z,v,w)$ has no physical meaning, but is an essential mathematical tool, introduced to reach a diffusion equation, and later, to derive the diffusion coefficient. After some mathematical developments [56,57,62], the second term on the left-hand side of Eq. (25) may be written as

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iiint v f_\varepsilon(x,z,v,w,t) dz\, dv\, dw$$
$$= -\frac{\partial}{\partial x} q(x,t) \iiint D^* v \phi(z,v,w) dz\, dv\, dw. \tag{28}$$

Introducing Eq. (28) in Eq. (25), the final expression of the macroscopic transport problem gives the one-dimensional diffusion equation (for the two-dimensional problem),

$$\frac{\partial}{\partial t} q(x,t) - \mathcal{K} \frac{\partial^2}{\partial x^2} q(x,t) = 0, \tag{29}$$

with the diffusion coefficient

$$\mathcal{K} = \iiint D^*(z,v,w) v\ \phi(z,v,w) dz\, dv\, dw, \tag{30a}$$

$$= \iiint D(z,v,w) v\ dz\, dv\, dw. \tag{30b}$$

The same analytical approach applied to the three-dimensional problem leads to the two-dimensional diffusion equation,

$$\frac{\partial}{\partial t} q(x,y,t) - \left(\mathcal{K}_{uu} \frac{\partial^2}{\partial x^2} + \mathcal{K}_{uv} \frac{\partial^2}{\partial x\, \partial y} + \mathcal{K}_{vu} \frac{\partial^2}{\partial y\, \partial x} + \mathcal{K}_{vv} \frac{\partial^2}{\partial y^2} \right) q(x,y,t) = 0, \tag{31}$$

with

$$\mathcal{K}_{uu} = \iiiint D_u(z,u,v,w) u\ dz\, du\, dv\, dw, \tag{32}$$

$$\mathcal{K}_{vv} = \iiiint D_v(z,u,v,w) v\ dz\, du\, dv\, dw, \tag{33}$$

$$\mathcal{K}_{uv} = \iiiint D_u(z,u,v,w) v\ dz\, du\, dv\, dw, \tag{34}$$

$$\mathcal{K}_{vu} = \iiiint D_v(z,u,v,w) u\ dz\, du\, dv\, dw, \tag{35}$$

where D_u and D_v are defined by

$$w \frac{\partial}{\partial z} D_u(z,u,v,w) = u\ \phi(z,u,v,w) \tag{36}$$

and

$$w \frac{\partial}{\partial z} D_v(z,u,v,w) = v\ \phi(z,u,v,w). \tag{37}$$

This asymptotic approach shows that the sound particle transport process, with complex boundaries conditions, can be reduced to a simple diffusion equation between the parallel planes, defined by only one parameter, the diffusion coefficient. From an acoustical point of view, this result suggests that the sound energy density along a street canyon with partially diffusely reflecting boundaries, is a solution of the diffusion equation. The diffuse behavior of the sound energy is then characterized by the diffusion coefficient, defined by Eq. (30a) in two dimensions (2D), or by Eqs. (32) and (35) in three dimensions (3D). This result is in agreement with previous work [63,64], but gives now a mathematical justification as well as an analytical expression for the diffusion coefficient. At this point, it is now necessary to determine the diffusion coefficient as a function of the boundary conditions. This is presented in the next paragraph.

5. The diffusion coefficient

In order to calculate the diffusion coefficient by Eq. (30a), the first step is to find the analytical expression of $\phi(z,v,w)$. Then, in the second step, an analytical expression of $D(z,v,w)$ can be given. In both cases, this requires to consider the real reflection law of the building façades defined by Eq. (13). However, the convergence of this asymptotic approach requires that k is greater than 1. As an example, the next developments are proposed for $k=2$, but the analytical expression of the diffusion coefficient can be found for any $k>1$.

a. Expression of the reflection law for $k=2$.

For $k=2$, the reflection law is written

$$\mathcal{R}(\mathbf{v},\mathbf{v}') = \mathcal{A}|w|^2, \tag{38}$$

where $\mathcal{A}$ is a normalization coefficient. Introducing the polar coordinates

$$v = c \cos \theta,$$

$$w = c \sin \theta,$$

$$dv\, dw = c\ d\theta, \tag{39}$$

the normalization condition gives,

$$\int_0^\pi \mathcal{R}(\theta,\theta') c\ d\theta = \int_0^\pi \mathcal{A}\ c^2 \sin^2 \theta\ c\ d\theta = 1 \tag{40}$$

leading to the expression of $\mathcal{A}$,

$$\mathcal{A} = \frac{2}{\pi c^3}. \tag{41}$$

Introducing Eq. (41) in Eq. (38) gives the reflection law, in Cartesian coordinates,

$$\mathcal{R}(\mathbf{v},\mathbf{v}') = \frac{2}{\pi c^3}|w|^2, \tag{42}$$

or, in polar coordinates,

$$\mathcal{R}(\theta,\theta') = \frac{2}{\pi c}\sin^2\theta. \tag{43}$$

b. Expression of $\phi(z,v,w)$.

To derive the diffusion coefficient, the function $\phi(z,v,w)$, must be expressed from the reflection laws at $z=+L$ and $-L$. In the limit where ε tends to zero, the function $\mathcal{F}$, defined by Eq. (20), must verify the boundary equation (16), leading to

$$|w|\phi^-(z,\mathbf{v})q(x,t) = d|w|\phi^+(z,\mathbf{v}^*)q(x,t) + (1-d)\int_{\Gamma^+}\mathcal{R}(\mathbf{v},\mathbf{v}') \times |w'|\phi^+(z,\mathbf{v})q(x,t)d\mathbf{v}', \tag{44}$$

where ϕ^+ (i.e., ϕ^-) is the restriction of ϕ to the set μ^+ (μ^-, respectively). Introducing Eq. (43) in Eq. (44), at both limits of the domain ($z=\pm L$), the function $\phi(z,v,w)$ must satisfy

$$\phi(L,v,w) = d\ \phi(L,v,-w) + (1-d)\frac{2}{\pi c}|w|\int_0^{\pi}|\sin\theta'|\phi(L,\theta')d\theta' \tag{45}$$

at $z=+L$ (i.e., $w<0$), and

$$\phi(-L,v,w) = d\ \phi(-L,v,-w) + (1-d)\frac{2}{\pi c}|w|\int_{\pi}^{2\pi}|\sin\theta'| \times \phi(-L,\theta')d\theta' \tag{46}$$

at $z=-L$ (i.e., $w>0$). Moreover, since the function $\phi(z,v,w)$ is independent of the space variable z, the two last relations lead to

$$\phi(L,v,-w)_{w<0} = \phi(-L,v,w)_{w>0} \tag{47}$$

and

$$\phi(-L,v,-w)_{w>0} = \phi(L,v,w)_{w<0}. \tag{48}$$

Introducing Eqs. (47) and (48) in the boundary conditions (45) and (46), respectively, at $z=+L$ and $z=-L$, gives for $w<0$ (i.e., $\pi<\theta<2\pi$),

$$\phi(L,v,w) = d\left(d\phi(-L,v,-w) + (1-d)\frac{2}{\pi c}|w|\int_{\pi}^{2\pi}|\sin\theta'| \times \phi(-L,\theta')d\theta'\right) + (1-d) \times \frac{2}{\pi c}|w|\int_0^{\pi}|\sin\theta'|\phi(L,\theta')d\theta', \tag{49}$$

and for $w>0$ (i.e., $0<\theta<\pi$),

$$\phi(-L,v,w) = d\left(d\phi(L,v,-w) + (1-d) \times \frac{2}{\pi c}|w|\int_0^{\pi}|\sin\theta'|\phi(L,\theta')d\theta'\right) + (1-d)\frac{2}{\pi c}|w|\int_{\pi}^{2\pi}|\sin\theta'|\phi(-L,\theta')d\theta'. \tag{50}$$

In both Eqs. (49) and (50), the integrals on the right-hand side are constants. Then, the function $\phi(z,v,w)$ is only proportional to the normal velocity component $|w|$. Separating the cases $w>0$ and $w<0$, the function $\phi(z,v,w)$ can be written as

$$\phi(z,v,w) = a_+|w| \quad \text{for } w>0 \tag{51}$$

and

$$\phi(z,v,w) = a_-|w| \quad \text{for } w<0, \tag{52}$$

where a_+ and a_- are two constants, which must be calculated. By introducing Eqs. (51) and (52) in the boundary conditions (45) and (46) gives, for $w<0$,

$$a_- = d\ a_+ + a_+(1-d)\frac{2}{\pi}\int_0^{\pi}\sin^2\theta' d\theta', \tag{53}$$

and, for $w>0$,

$$a_+ = d\ a_- + a_-(1-d)\frac{2}{\pi}\int_{\pi}^{2\pi}\sin^2\theta' d\theta'. \tag{54}$$

This system of equations has only one solution, leading to $a_+=a_-$. Finally, the function $\phi(z,v,w)$ can be written as

$$\phi(z,v,w) = a_+|w|. \tag{55}$$

The last equations can be generalized to $|w|^k$ reflection laws, leading to

$$\phi(z,v,w) = a_+|w|^{k-1} \quad \text{for } w>0 \tag{56}$$

and

$$\phi(z,v,w) = a_+|w|^{k-1} \quad \text{for } w<0. \tag{57}$$

The constant a_+ can be found by using the normalization condition of the function $\phi(z,v,w)$ given by Eq. (21), which can now be written (for $k=2$) as

$$\int_{-L}^{L}\left(\int_{0}^{\pi}|\sin\theta|d\theta+\int_{\pi}^{2\pi}|\sin\theta|d\theta\right)dz=\frac{1}{c^2a_+} \tag{58}$$

leading to

$$a_+=\frac{1}{8\,L\,c^2}. \tag{59}$$

c. Expression of $D(z,v,w)$.

The second step of the calculation of the diffusion coefficient is the determination of the function $D(z,v,w)$, defined by Eq. (27) in Cartesian coordinates, or by the following relation in polar coordinates:

$$\sin\theta\frac{\partial}{\partial z}D(z,\theta)=\frac{\cos\theta}{8\,L\,c}|\sin\theta| \tag{60a}$$

$$=\mathcal{C}\cos\theta|\sin\theta|, \tag{60b}$$

with the constant parameter $\mathcal{C}=1/8\,L\,c$. By separating the two cases $w>0$ and $w<0$, Eq. (60b) gives

$$D(z,\theta)=\mathcal{C}z\cos\theta+\mathcal{A}_+(\theta),\quad \text{for } w>0 \tag{61}$$

and

$$D(z,\theta)=-\mathcal{C}z\cos\theta+\mathcal{A}_-(\theta),\quad \text{for } w<0, \tag{62}$$

where $\mathcal{A}_+(\theta)$ and $\mathcal{A}_-(\theta)$ are constants of integration. The function $D(z,\theta)$ must verify the boundary conditions

$$D^-(z,\theta)=dD^+(z,2\pi-\theta)+(1-d)\int_{\Gamma^+}\mathcal{R}(\theta,\theta')D^+(z,\theta')cd\theta', \tag{63}$$

that gives at $z=+L$ (i.e., $w<0$),

$$D(L,\theta)=dD(L,2\pi-\theta)+(1-d)\frac{2}{\pi}\sin^2\theta\int_0^{\pi}D(L,\theta')d\theta', \tag{64}$$

and at $z=-L$ (i.e., $w>0$),

$$D(-L,\theta)=dD(-L,2\pi-\theta)+(1-d)\frac{2}{\pi}\sin^2\theta\times\int_{\pi}^{2\pi}D(-L,\theta')d\theta'. \tag{65}$$

Introducing Eqs. (61) and (62), in the boundary conditions (64) yields for $w<0$,

$$-\mathcal{C}L\cos\theta+\mathcal{A}_-(\theta)=d[\mathcal{C}L\cos\theta+\mathcal{A}_+(\theta)]+(1-d)\frac{2}{\pi}\sin^2\theta\int_0^{\pi}[\mathcal{C}L\cos\theta'+\mathcal{A}_+(\theta')]d\theta', \tag{66}$$

and gives

$$\mathcal{A}_-(\theta)=\mathcal{C}L(1+d)\cos\theta+d\mathcal{A}_+(\theta)+C_1\sin^2\theta, \tag{67}$$

where C_1 is a constant equal to

$$C_1=(1-d)\frac{2}{\pi}\int_0^{\pi}[\mathcal{C}L\cos\theta'+\mathcal{A}_+(\theta')]d\theta'. \tag{68}$$

Moreover, introducing Eqs. (61) and (62), in the boundary condition (65) yields for $w>0$,

$$-\mathcal{C}L\cos\theta+\mathcal{A}_+(\theta)=d[\mathcal{C}L\cos\theta+\mathcal{A}_-(\theta)]+(1-d)\frac{2}{\pi}\sin^2\theta\int_{\pi}^{2\pi}[\mathcal{C}L\cos\theta'+\mathcal{A}_-(\theta')]d\theta', \tag{69}$$

leading to

$$\mathcal{A}_+(\theta)=\mathcal{C}L(1+d)\cos\theta+d\mathcal{A}_-(\theta)+C_2\sin^2\theta, \tag{70}$$

where C_2 is a constant defined by

$$C_2=(1-d)\frac{2}{\pi}\int_{\pi}^{2\pi}[\mathcal{C}L\cos\theta'+\mathcal{A}_-(\theta')]d\theta'. \tag{71}$$

The parameter $\mathcal{A}_+$ is found by introducing Eq. (67) in Eq. (70) for $w>0$,

$$\mathcal{A}_+(\theta)=\mathcal{C}L\frac{1+d}{1-d}\cos\theta+C_3\sin^2\theta, \tag{72}$$

where C_3 is a constant equal to

$$C_3=\frac{dC_1+C_2}{1-d^2}. \tag{73}$$

In the same way, for $w<0$,

$$\mathcal{A}_-(\theta)=\mathcal{C}L\frac{1+d}{1-d}\cos\theta+C_4\sin^2\theta, \tag{74}$$

where C_4 is a constant equal to

$$C_4=\frac{dC_2+C_1}{1-d^2}. \tag{75}$$

Thus, introducing Eq. (72) in Eq. (61) and Eq. (74) in Eq. (62), the function $D(z,\theta)$ must satisfy

$$D(z,\theta)=\mathcal{C}\left(z+L\frac{1+d}{1-d}\right)\cos\theta+C_3\sin^2\theta, \tag{76}$$

for $0<\theta<\pi$ (i.e., $w>0$), and

TABLE I. Two-dimensional (a) and three-dimensional (b) expressions of the diffusion coefficient (in m^2/s), in a street of width $2L$, for building façades defined by $|w|^k$ diffuse reflection laws. d and c are, respectively, the accommodation coefficient of the building façades and the sound velocity (in m/s).

(a)

k	Two-dimensional diffusion coefficient
$2q$ (for $q>1$)	$\pi/4^q q(2q-1)((q-1)!)^2[\prod_{l=1}^{q-1}(2l+1)]^2 K$
$2q+1$ (for $q>1$)	$q4^q/\pi(2q+1)[\prod_{l=1}^{q-1} l/(2l+1)]^2 K$

(b)

k	Three-dimensional diffusion coefficient
$k>1$	$K\cdot[k/(k^2-1)]$

$$D(z,\theta)=\mathcal{C}\left(-z+L\frac{1+d}{1-d}\right)\cos\theta+C_4\sin^2\theta, \qquad (77)$$

for $\pi<\theta<2\pi$ (i.e., $w<0$).

d. Expression of the diffusion coefficient.

The last step is the derivation of the diffusion coefficient using the relation (30a), in polar coordinates,

$$\mathcal{K}=\iint c^2\cos\theta D(z,\theta)dz\,d\theta. \qquad (78)$$

Separating the cases $0<\theta<\pi$ and $\pi<\theta<2\pi$, Eq. (78) becomes

$$\mathcal{K}=c^2\int_{-L}^{L}\int_{0}^{\pi}\cos\theta D(z,\theta)dz\,d\theta + c^2\int_{-L}^{L}\int_{\pi}^{2\pi}\cos\theta D(z,\theta)dz\,d\theta. \qquad (79)$$

Introducing Eqs. (76) and (77) in Eq. (79), the diffusion coefficient for $k=2$, is written

$$\mathcal{K}=\frac{\pi}{4}\frac{1+d}{1-d}Lc=\frac{\pi}{4}K, \qquad (80)$$

with

$$K=\frac{1+d}{1-d}Lc. \qquad (81)$$

The same developments can also be derived in three dimensions, leading to $K_{uu}=K_{vv}=\mathcal{K}$ and $K_{uv}=K_{vu}=0$, due to the axisymmetry of the reflection laws. For $k=2$, the diffusion coefficient is given by

$$\mathcal{K}=\frac{2}{3}\frac{1+d}{1-d}Lc=\frac{2}{3}K. \qquad (82)$$

These expressions of the diffusion coefficient can easily be generalized to any value of $k>1$, by using the same methods. All expressions are given both in two and three dimensions in Table I.

Table I shows that the diffusion coefficient depends on the morphology of the building façades. First, in order to calculate the diffusion coefficient, the analytical form of the reflection law must be introduced. For $|w|^k$ reflection laws, Fig. 9 shows that the diffusion coefficient, decreases with k. As expected, when the reflection law concentrates the sound energy around the normal to the building façades (increasing k), the diffusion of sound energy is less important. Second, as shown by Fig. 10, K increases with the accommodation coefficient d, meaning that the diffusion of the sound energy will be faster. In other words, the sound energy is concentrated for a longer time in the street and around the sound source in the case of diffusely reflecting building façades than in the case of specular reflecting façades. Finally the diffusion coefficient also increases with the street width (Fig. 10). For small street width, the sound energy remains a longer time around the sound source.

D. Sound energy in a street canyon

The previous developments have shown that the sound energy along the median plane of the street is solution of the diffusion equation (29) in 2D or (31) in 3D, with the diffusion coefficients given in Table I. Conversely, the sound energy in a transversal line of the street (along z) is constant.

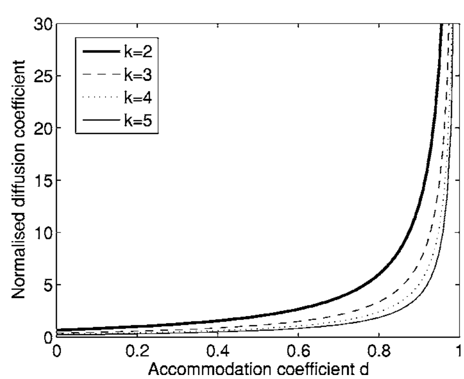

FIG. 9. Three-dimensional normalized diffusion coefficient $\mathcal{K}/Lc$ in s, from (b) of Table I, as a function of the accommodation coefficient d, and for several values of k.

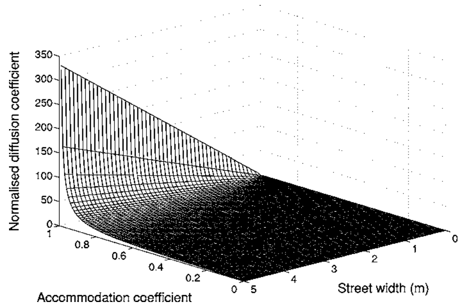

FIG. 10. Three-dimensional normalized diffusion coefficient $\mathcal{K}/c$, from (b) of Table I, as a function of the accommodation coefficient d and the street width $2L$ (for $k=2$).

1. Infinite solution

Considering an impulsive sound source between two infinite planes, the solution of the diffusion equation is then simply

$$q(x,y,t) = \frac{1}{4\pi\mathcal{K}t}\exp\left(-\frac{x^2+y^2}{4\mathcal{K}t}\right), \tag{83}$$

which is the Green function of the diffusion equation. In this expression, the origin of the coordinates is taken at the sound source, located in the middle of the median plane.

2. Finite solution

However, in the following numerical simulations (see Sec. V) as well as in real cases, the street is not infinite. Openings at the top and both extremities of the street produce an extra attenuation, due to sound particles absorption, that must be introduced in the model. In this case, the propagation medium is not defined by two infinite planes, but is delimited by two finite planes, with a length l_x (i.e., the street length) and height $L_y=2l_y$ (twice the street height) due to the mirror plane created by the pavement (Fig. 4). Four boundary equations must be considered, one for each extremity of the propagation medium.

Since it is too difficult to take this absorption into account in the boundary conditions (12) of the transport equation, the choice was made to introduce it in the boundary conditions of the diffusion equation. As in heat transfer problems, an exchange coefficient h is introduced. It characterizes the energy exchanges between the exterior and the interior of the domain (i.e., the sound particles flow), leading to the following equations:

$$\pm\mathcal{K}\frac{\partial q(x,y,t)}{\partial x} = h\, q(x,y,t) \quad \text{for } x=0 \text{ and } l_x,\ \forall\, y, \tag{84}$$

$$\pm\mathcal{K}\frac{\partial q(x,y,t)}{\partial y} = h\, q(x,y,t) \quad \text{for } y=0 \text{ and } L_y,\ \forall\, x.$$

In the last boundary equations, the origin of the coordinates is located on top of the street [Fig. 4(a)], while the sound source is still on the pavement ($z=0$), in the middle of the street (i.e., the middle of the median plane) at $(x,y)=(l_x/2,l_y)=(l_x/2,L_y/2)$.

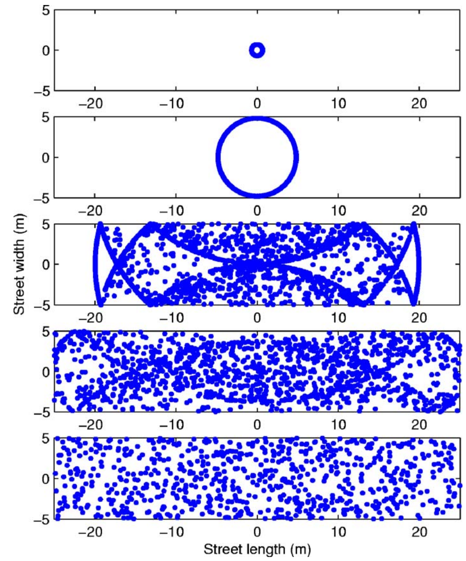

FIG. 11. (Color online) Example of two-dimensional propagation of 2000 sound particles in a street ($2L=10$ m, $|w|^2$ reflection law with $d=0.5$) after 4, 16, 60, 100 and 200 ms. The source is located in the middle of the street at position (0, 0).

a. Expression of the exchange coefficient. The exchange coefficient h is defined by writing the flow $J(x,y,z,t)$ of sound particles leaving the propagation medium. At first, considering the definition given in Sec. II C for the transport phenomena, the sound particles flow can be written as

$$J_t(x,y,z,t) = \int_{\Gamma^+} |\mathbf{v}\cdot\mathbf{n}| f(x,y,z,u,v,w,t)\,du\,dv\,dw, \tag{85}$$

for $(x,y)\in\Omega$, $\forall z$ and where $\mathbf{n}$ is the exterior normal to the street openings. Since the distribution function $f(x,y,z,u,v,w,t)$ is the product of two functions $q(x,y,t)$ and $\phi(z,u,v,w)$, the last equation (85) gives, for example, at $x=l_x$,

$$J_t = q(l_x,y,t)\int_{\Gamma^+} u\ \phi(z,u,v,w)\,du\,dv\,dw. \tag{86}$$

Moreover, according to the definition of the sound particle flow (84) in the diffusion behavior, it can also be written as

$$J_d = -\mathcal{K}\frac{\partial q(l_x,y,t)}{\partial x}\int_{\Gamma^+}\phi(z,u,v,w)\,du\,dv\,dw, \tag{87}$$

$$= h\ q(l_x,y,t)\int_{\Gamma^+}\phi(z,u,v,w)\,du\,dv\,dw. \tag{88}$$

Then, the diffusion behavior is in agreement with the transport phenomena, with the condition that $J_t=J_d$, leading to

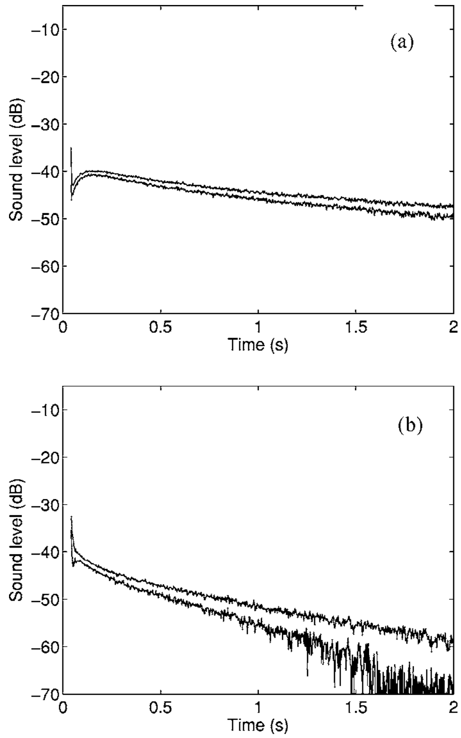

FIG. 12. Standard deviation (from the mean value) of the distribution of the sound particles along a transversal line between the two planes, for $2L=4$ m with the $|w|^2$ law, at 15 m from the sound source, and for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

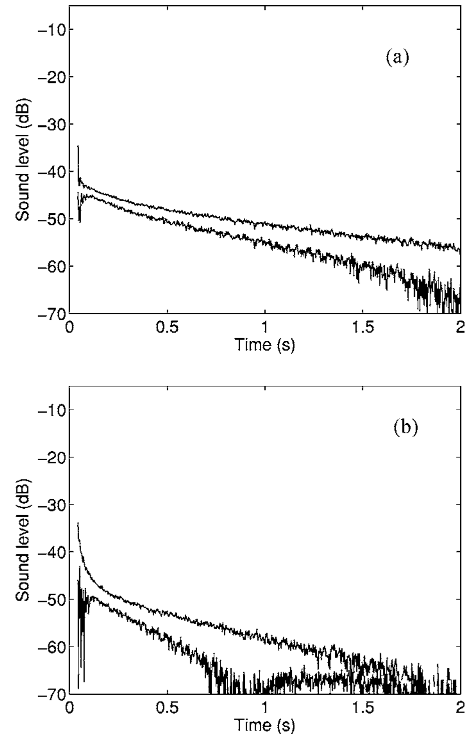

FIG. 13. Standard deviation (from the mean value) of the distribution of the sound particles along a transversal line between the two planes, for $2L=10$ m with the $|w|^2$ law, at 15 m from the sound source, and for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

$$q(l_x,y,t)\int_{\Gamma^+} u\ \phi(z,u,v,w)du\ dv\ dw$$
$$= h\ q(l_x,y,t)\int_{\Gamma^+} \phi(z,u,v,w)du\ dv\ dw. \quad (89)$$

For a w^k reflection law, the exchange coefficient can be derived. Introducing Eq. (51) in Eq. (89), the exchange coefficient h must verify

$$h\int_{\Gamma^+} |w|^{k-1}du\ dv\ dw = \int_{\Gamma^+} u|w|^{k-1}du\ dv\ dw. \quad (90)$$

The integration of both integrals in Eq. (90) gives the following expression for the exchange coefficient:

$$h = \frac{kc}{\pi}\int_0^{\pi} \sin^2\theta|\cos\theta|^{k-1}d\theta, \quad (91)$$

leading, for example, to $h=4c/3\pi$ for $k=2$, and $h=3c/8$ for $k=3$. Considering the same development at $x=0$, $y=0$, and $y=L_y$ will give the same expression for h. As expected, the expression of h is independent of the size of the extremities, but is a function of the reflection law only.

b. Finite solution of the diffusion equation.

The solution of diffusion equation for a finite medium with the boundary equations (84) is then [65],

$$q(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n}{u_n}\frac{b_m}{v_m}\left[u_n\cos\left(\frac{u_n x}{l_x}\right) + B_x\sin\left(\frac{u_n x}{l_x}\right)\right] \times\left[v_m\cos\left(\frac{v_m y}{L_y}\right) + B_y\sin\left(\frac{v_m y}{L_y}\right)\right]\exp\left(-\frac{u_n^2}{l_x^2} - \frac{v_m^2}{L_y^2}\mathcal{K}t\right), \quad (92)$$

where the u_n and v_m are coefficients given by solving the following equations:

$$(u_n^2 - B_x^2)\tan u_n = 2B_x u_n \quad (93)$$

and

$$(v_m^2 - B_y^2)\tan v_m = 2B_y v_m, \quad (94)$$

with $B_x=hl_x/\mathcal{K}$ and $B_y=hL_y/\mathcal{K}$. For a point source at $(x,y)=(l_x/2,L_y/2)$, a_n/u_n, and b_m/v_m are given by

$$\frac{a_n}{u_n} = \frac{2}{l_x}\frac{u_n\cos(u_n/2) + B_x\sin(u_n/2)}{u_n^2 + B_x^2 + 2B_x} \quad (95)$$

and

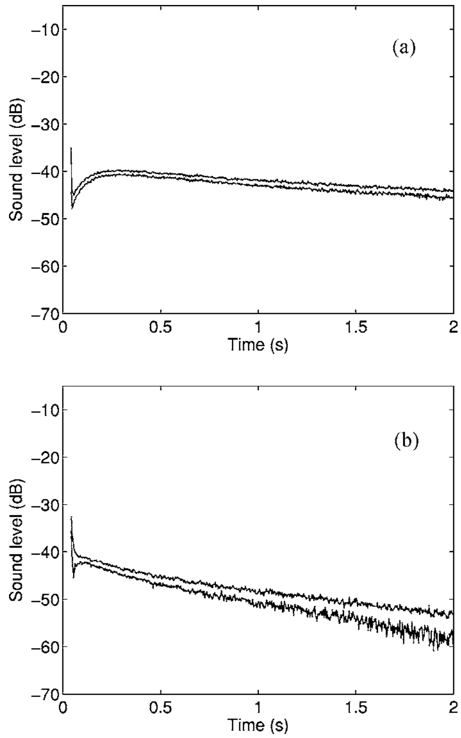

FIG. 14. Standard deviation (from the mean value) of the distribution of the sound particles along a transversal line between the two planes, for $2L=4$ m with the $|w|^4$ law, at 15 m from the sound source, and for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

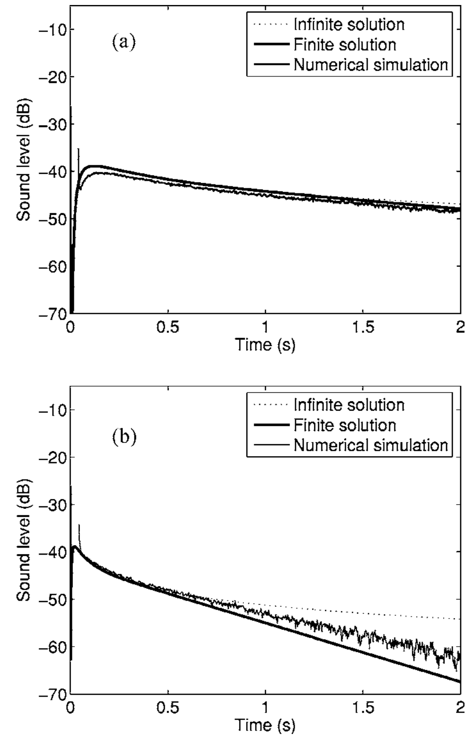

FIG. 15. Comparison between the analytical solutions and the numerical distribution of the sound particles, for $2L=4$ m with the $|w|^2$ law, at 15 m from the sound source, for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

$$\frac{b_m}{v_m} = \frac{2}{L_y}\frac{v_m\cos(v_m/2) + B_y\sin(v_m/2)}{v_m^2 + B_y^2 + 2B_y}. \tag{96}$$

In practice, Eqs. (93) and (94) must be numerically solved in order to determine the u_n and v_m coefficients. These coefficients are then introduced in Eqs. (95) and (96) to calculate the a_n and b_m coefficients. Finally, the finite solution of the diffusion equation is computed by considering the first elements of Eq. (92).

V. NUMERICAL VALIDATION

A. Introduction

Numerical simulations have been performed to validate the model for a street canyon. Simulations have been carried out to test the validity and the sensitivity of the asymptotic approach. Indeed, the asymptotic development has been realized for a street width tending towards zero. Thus, it is necessary to test the convergence of the solution and the limit of the asymptotic approach when the street width increases to realistic values.

B. Principle of the Monte Carlo simulations

The principle of the numerical simulation consists in following the path of sound particles between two parallel planes (as an infinite street) numerically. Each sound particle propagates in a straight line between two successive collisions with the parallel planes. At each collision, the new direction of the sound particle is found by generating a random direction in agreement with the reflection law of the building façades.

At first, considering an omnidirectional sound source located between the planes, N sound particles are emitted in random directions. This is done by generating, for each particle, two random angles, θ and φ, between 0 and 2π, and $-\pi/2$ and $\pi/2$, respectively. Then, at each sound particle collision on a plane, a first random number ξ is generated and compared to the accommodation coefficient d of the plane. If $\xi \geqslant d$, the sound particle is reflected in the specular direction. Inversely, if $\xi < d$, the sound particle is reflected with a diffuse reflection, according to the diffuse reflection law of the building façades. For the $|w|^k$ reflection law, the reflection direction is found by using the rejection method [66]. If the sound particle crosses the openings, it disappears from the simulations.

The exact position of the sound particles is calculated for each time increment Δt. A rectangular meshing $\Delta x \times \Delta y \times \Delta z$ is carried out between the planes to follow the sound particles distribution in the medium as a function of time. The choice of N, Δt, Δx, Δy, and Δz is a compromise between the calculation time of the numerical simulations and the temporal and spatial accuracy of the results.

As an example, Fig. 11 shows the numerical results of the two-dimensional sound propagation of 2000 sound particles in a street. Similar simulations have also been realized in three-dimensional streets (i.e., between two planes).

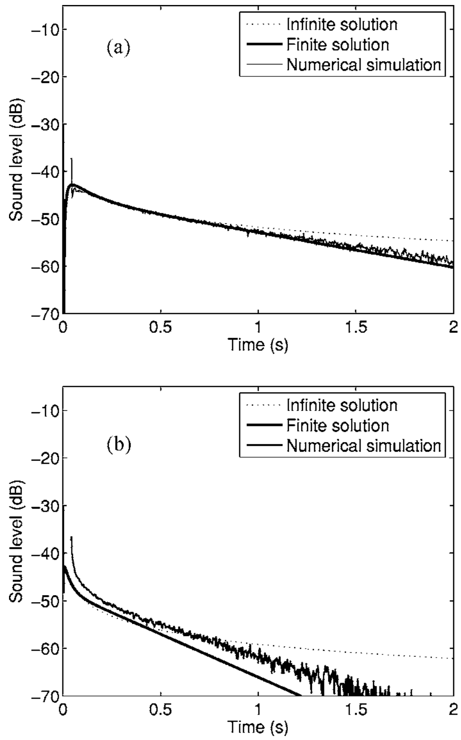

FIG. 16. Comparison between the analytical solutions and the numerical distribution of the sound particles, for $2L=10$ m with the $|w|^2$ law, at 15 m from the sound source, for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

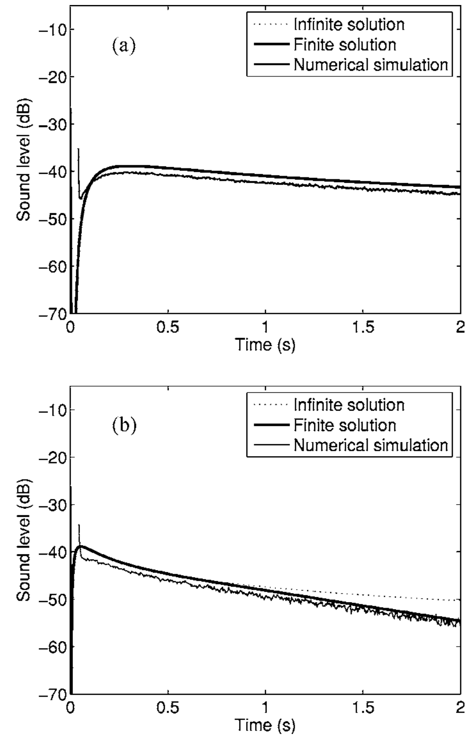

FIG. 17. Comparison between the analytical solution and the numerical distribution of the sound particles, for $2L=4$ m with the $|w|^4$ law, at 15 m from the sound source, for two accommodation coefficients, (a) $d=0$ and (b) $d=0.7$.

C. Validation

1. Uniformity along the z axis

Both in two and three dimensions, the mathematical development has shown that the distribution of sound particles along the z axis is uniform, while the distribution along the median line (2D) or plane (3D) follows a diffusion process. The first validation is then to verify this uniformity. Three-dimensional numerical simulations have been performed for several configurations of street width ($2L=4$, 7, and 10 m), reflection law ($|w|^2$ and $|w|^4$) and accommodation coefficient ($d=0$ and 0.7), using 10^6 sound particles. Since the street is modelled as two planes with a sound source in the middle of the median plane, results should be equivalent along the four semiaxis of the median plane ($\pm x$ and $\pm y$). Therefore, numerical simulations have been averaged along these semiaxes in order to avoid small numerical fluctuations and to increase the accuracy of the numerical results.

In order to evaluate the uniformity of the sound particles distribution along the z axis, the mean value $\bar{n}$ and the standard deviation σ of the number of sound particles were estimated for receivers along the z axis, at several distances from the sound source and at each time step. As an example, Fig. 12 shows the standard deviation from the mean value (i.e., $\bar{n}-\sigma$ and $\bar{n}+\sigma$) of the number of sound particles with time, for a receiver located at 15 m from the source, in a street of width $2L=4$ m with the $|w|^2$ reflection law, and for two accommodation coefficients. For $d=0$ [Fig. 12(a), i.e., reflections are fully diffuse], the standard deviation is small, suggesting that the distribution of sound particles along the z axis is in agreement with the theoretical model. When the accommodation coefficient, increases [Fig. 12(b)], reflections become more specular and the standard deviation increases principally for larger propagation time. However, numerical results are still in agreement with the model. Similar conclusions can also be given for receivers closer or further from the sound source. For larger streets (Fig. 13), the standard deviation increases. For $d=0$ [Fig. 13(a)], results are still satisfactory, whereas for $d=0.7$ [Fig. 13(b)], the distribution of sound particles becomes less uniform. As expected, the analytical model is well adapted for smaller street. Finally, one can also remark in Fig. 14 that there is more uniformity when the reflection law is more distributed around the normal to the façades.

2. Comparison with the analytical solutions

In addition, Figs. 15,16 and 17 show a comparison between both infinite and finite solutions, Eqs. (83) and (92), and the mean value of the number of sound particles with time, for a receiver located at 15 m from the source, for various street width ($2L=4$ m and $2L=10$ m) and reflection laws ($|w|^2$ and $|w|^4$), and for two accommodation coefficients ($d=0$ and $d=0.7$). As expected with the previous results, good agreement is observed between the model (particularly the finite solution) and the numerical simulations, first for small accommodation coefficients [for example, see Figs. 15(a) and 15(b)], and second when the reflection law is more concentrated around the normal to the façades (for example, see Figs. 15 and 16). However, when the reflection is more specular [$d=0.7$ in Figs. 15(a) and 16(a), for example], the model is less in agreement. As seen in Figs. 15(a) and 16(a), the model is in better agreement with numerical values for smaller street width, as expected by the asymptotic approach. Similar conclusions can also be given for receivers closer or further from the sound source.

VI. CONCLUSION

In this paper, a general formulation based on the transport of sound particles is proposed for sound field modeling in architectural acoustics. By using this approach, the spatial and temporal evolution of the sound field in an enclosure with complex boundary conditions can be analytically expressed. For street canyons, the transport equation can be reduced to a diffusion equation, whose diffusion coefficient is defined from the street size and the reflection law of the building façades.

The analytical model has been compared to numerical simulations of sound particles propagation in street canyon, for several reflection laws of the building façades, accommodation coefficient and street width. As expected by the choice of the asymptotic approach, the model is in agreement with numerical simulations, especially for smallest street widths, and when diffuse reflections are concentrated around the normal to the building façades. However, when the street width increases or when the reflection law is close to a specular reflection, the diffusion model fails.

Moreover, the finite solution, including the sound particles absorption at the open top and both extremities of the street, gives better agreement with the numerical solution than the infinite solution. When the street width decreases and when diffuse reflections are concentrated around the normal, both infinite and finite solutions give equivalent results.

This approach is an interesting way to model the sound field in architectural acoustics, especially when the limits of the domain produce both specular and diffuse reflections. In the present model, diffuse reflections can be taken into account by considering any kind of reflection law, while most of current analytical and numerical models consider Lambert's law of reflection only. Although models with Lambert's law can give satisfactory results, recent studies seem to show that the choice of the reflection law can influence the sound attenuation and the reverberation in an enclosure or in a street [67]. Therefore, other approaches must be considered, as the one presented in this paper.

At this stage, and in order to have a more useful model, the sound absorption by the building façades, the pavement, and the openings should now be included in the approach. On the other hand, two different reflection laws on both sides of the street should be also considered. In the long term, by using this mathematical approach, many others developments can also be introduced. Thus, by considering a scattering kernel, it could be possible to take the multiple scattering by diffusers filling an enclosure into account. Meteorological effects can also be analytically included in the model. For example, wind, whose effect can be important in urban areas, could be introduced by considering a transport force in the transport equation. Atmospheric attenuation can be included by considering a new probabilistic term, which would express the probability that a sound particle may disappear during its propagation. Last, complex configurations with atmospheric turbulence could be studied by considering the coordinate dependence of the sound particle velocity in the derivation of the transport equation and in its resolution.

[1] R. H. Lyon, J. Acoust. Soc. Am. **55**, 493 (1974).
[2] M. Hodgson, J. Acoust. Soc. Am. **89**, 765 (1991).
[3] J. Kang, J. Acoust. Soc. Am. **107**, 1394 (2000).
[4] R. R. Torres, Acust. Acta Acust. **86**, 919 (2000).
[5] H. G. Davies and R. H. Lyon, J. Acoust. Soc. Am. **54**, 1565 (1973).
[6] R. Bullen and F. Fricke, J. Sound Vib. **54**, 123 (1977).
[7] R. Bullen and F. Fricke, J. Sound Vib. **46**, 33 (1976).
[8] V. Markovic, Acust. Acta Acust. **84**, 570 (1998).
[9] B. I. Dalenbäck, M. Kleiner, and P. Svensson, J. Audio Eng. Soc. **42**, 793 (1994).
[10] H. Kuttruff, *Room Acoustics* (Applied Science Publishers, London, 1973).
[11] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).
[12] L. Cremer and H. A. Müller, *Principle and Applications of Room Acoustics* (Applied Science Publishers, London, New York, 1982), Vol. 1.
[13] M. Hodgson, J. Acoust. Soc. Am. **84**, 253 (1988).
[14] M. Hodgson, J. Acoust. Soc. Am. **94**, 835 (1993).
[15] J. B. Allen and D. A. Berkley, J. Acoust. Soc. Am. **65**, 943 (1979).
[16] H. Juricic and F. Santon, Acustica **28**, 77 (1973).
[17] J. Borish, J. Acoust. Soc. Am. **75**, 1827 (1984).
[18] D. G. Holmes and R. H. Lyon, in *Second Interagency Symposium on University Research in Transportation Noise* (North Carolina State University, Raleigh, 1974).
[19] M. Vorländer, J. Acoust. Soc. Am. **86**, 172 (1989).
[20] A. Kulowski, Appl. Acoust. **18**, 449 (1985).
[21] A. Krokstad, S. Strom, and S. Sorsdal, J. Sound Vib. **8**, 118 (1968).
[22] T. Lewers, Appl. Acoust. **38**, 161 (1993).
[23] H. Kuttruff, Acustica **25**, 333 (1971) (in German).
[24] H. Kuttruff, Acustica **35**, 141 (1976) (in German).
[25] M. M. Carroll and R. N. Miles, J. Acoust. Soc. Am. **64**, 1424 (1978).
[26] H. Kuttruff and T. Straßen, Acustica **45**, 246 (1980) (in German).
[27] T. Hidaka, in *CIARM'95 2nd International Conference on Acoustics and Musical Research* (CIARM, Ferrara, 1995), pp. 11–22.
[28] J. J. Embrechts, D. Archambeau, and G. Stan, Acust. Acta

Acust. **87**, 482 (2001).
[29] H. Kuttruff, Acustica **18**, 131 (1967) (in German).
[30] E. A. Lindqvist, Acustica **50**, 313 (1982).
[31] E. A. Lindqvist, Appl. Acoust. **16**, 183 (1983).
[32] R. Bullen and F. Fricke, J. Sound Vib. **80**, 11 (1982).
[33] K. Li and M. Hodgson, J. Sound Vib. **218**, 463 (1998).
[34] U. J. Kurze, J. Sound Vib. **98**, 349 (1985).
[35] H. Imaizumi, S. Kunimatsu, and T. Isei, J. Acoust. Soc. Am. **108**, 632 (2000).
[36] J. Kang, Acustica **82**, 509 (1996).
[37] H. G. Davies, J. Acoust. Soc. Am. **53**, 1253 (1971).
[38] H. Kuttruff, J. Acoust. Soc. Am. **69**, 1716 (1981).
[39] H. Nélisse and J. Nicolas, J. Acoust. Soc. Am. **101**, 3517 (1997).
[40] A. M. Ondet and J. L. Barbry, J. Acoust. Soc. Am. **85**, 787 (1989).
[41] S. M. Dance, Appl. Acoust. **63**, 359 (2002).
[42] H. G. Davies, J. Acoust. Soc. Am. **64**, 517 (1978).
[43] P. R. Donavan, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, 1976.
[44] W. B. Joyce, J. Acoust. Soc. Am. **58**, 643 (1975).
[45] J. L. B. Coelho, Acust. Acta Acust. **86**, 903 (2000).
[46] W. B. Joyce, Phys. Rev. D **9**, 3234 (1974).
[47] L. de Broglie, *La thermodynamique de la Particule Isolée* (Gauthiers-Villars, Paris, 1964).
[48] R. Hakim, *Mécanique* (Armand Colin, Paris, 1995).
[49] M. M. R. Williams, *Mathematical Methods in Particle Transport Theory* (Butterworth, London, 1971).
[50] A. M. Weinberg and E. P. Wigner, *The Physical Theory of Neutron Chain Reactors* (The University of Chicago Press, Chicago, IL, 1958).
[51] H. L. Frisch, Phys. Rev. **107**, 917 (1957).
[52] I. Prigogine, *Non-Equilibrum Statistical Mechanics*, (Intersciences, New York, 1962), Vol. 1
[53] Y. W. Lam, J. Acoust. Soc. Am. **105**, 762 (1999).
[54] M. Vorländer and E. Mommertz, Appl. Acoust. **60**, 187 (2000).
[55] C. Cercignani, *Theory and Application of the Boltzmann Equation* (Academic, London, 1975).
[56] H. Babovsky, C. Bardos, and T. Platkowski, Asymptotic Anal. **3**, 265 (1991).
[57] C. Börgers, C. Greengard, and E. Thomann, SIAM J. Appl. Math. **52**, 1057 (1992).
[58] D. Gaulin, Ph.D. thesis, Université du Maine, 2000.
[59] J. Kang, Acust. Acta Acust. **88**, 77 (2002).
[60] D. Oldham and M. Radwan, Build. Acoust. **1**, 65 (1994).
[61] J. Picaut and M. Bérengier, INTERNOISE 2001, The 2001 International Congress and Exhibition on Noise Control Engineering, The Hague, Holland, August 2001.
[62] T. Le Pollès, Ph.D. thesis, Université du Maine, 2003.
[63] J. Picaut, J. Hardy, and L. Simon, Phys. Rev. E **60**, 4851 (1999).
[64] J. Picaut, J. Hardy, and L. Simon, J. Acoust. Soc. Am. **106**, 2638 (1999).
[65] J. Picaut, L. Simon, and J.-D. Polack, Appl. Acoust. **56**, 217 (1999).
[66] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Mannery, *Numerical Recipes in FORTRAN 77, the Art of Scientific Computing* (Cambridge University Press, Cambridge 1992).
[67] J. Picaut, INTERNOISE 2004, The 33rd International Congress and Exposition on Noise Control Engineering, Prague, Czech Republic, August 2004.